Basic Concepts of Differential Geometry and Fibre Bundles

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ABSTRACT

The concept of a manifold is central to many parts of geometry and modern mathematical physics because it allows more complicated structures to be described and understood in terms of the relatively well-understood properties of Euclidean space. A manifold is roughly a continuous topological space which is locally similar to Euclidean space but which need not be Euclidean globally. Fibre bundle is a very interesting manifold and is formed by combining a manifold M with all its tangent spaces. A fibre bundle is a manifold that looks locally like a product of two manifolds, but is not necessarily a product globally. In this study some definitions are given to make the study easier to the common readers. An attempt has taken here to discuss elementary ideas of manifolds and fibre bundles in an easier way.

Keywords: Manifold, Fibre bundles, Möbius band, Tangent space, Orientation

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INTRODUCTION

In the 20th century the discussion of physical concepts is sometimes based on the properties of gauge theories, topology and differential geometry (differentiable manifolds and fibre bundles). Differential geometry discusses curves, surfaces, length, volume, and curvature using the methods of calculus. In physics, the manifold may be the space-time continuum and the bundles and connections are related to various physical fields. Differential geometry is used in Einstein's general theory of relativity. According to this theory, the universe is a smooth manifold equipped with a pseudo-Riemannian metric, which describes the curvature of space-time. Differential geometry is also used in the study of gravitational focusing and black holes. Differential geometry has applications to both Lagrangian mechanics and Hamiltonian mechanics.

The term "manifold" comes from German Mannigfaltigkeit, by Bernhard Riemann. In English, "manifold" refers to spaces with a differentiable or topological structure.

A manifold is roughly a continuous topological space which is locally similar to Euclidean space but which need not be Euclidean globally. A differentiable manifold *M* is said to be smooth if it is infinitely differentiable. Lines and circles are 1-dimensional manifolds; surfaces are 2-dimensional manifolds, plane and sphere are 3-dimensional manifolds and Lorentzian space-time manifolds in general relativity are 4-dimentional.

In Euclidean geometry all points of R^n can be covered by one coordinate frame $\left(x^1,\ldots,x^n\right)$ and all frames with such a property are related to each other by general a linear transformation, that is, by the elements of the general linear group $GL(n,R)$ as (Frè 2013):

$$x'^{\mu} = a^{\mu}_{\nu} x^{\nu}\text{, where } a^{\mu}_{\nu} = GL(n,R).$$

The space-time manifold is the flat affine manifold R^4 both in Galileo transformations and Lorentz transformations with Galilei or Lorentz subgroups of $GL(4,R)$. But a different situation arises when the space-time manifold is non-flat. In such a situation we cannot express all points of a curved surface in a single coordinate frame, i.e., in a single chart. We can introduce this curved surface by a collection of charts, called atlas, each of which maps one open region of the surface such that the union of all these regions covers the entire surface. The concept of an atlas of open charts, suitably reformulated in mathematical terms, provides the definition of a differentiable manifold, for more complicated non-flat situations.

SOME RELATED DEFINITIONS

In this section we provide some definitions following Hawking and Ellis (1973), Joshi (1996) and Mohajan (2013b), which are fully related to the discussion of this study.

Open Set

Any point *p* contained in a set *S* can be surrounded by an open sphere or ball $|x-p|<r$, all of whose points lie entirely in *S*, where $r>0$; usually it is denoted by;

$$S(p,r)=\{\,x:d(p,x)<r\}.$$

Closed Set

A subset *S* of a topological space *M* is a closed set iff its complement S^c is an open set.

Topological Space

Let *M* be a non-empty set. A class *T* of subsets of *M* is a topology on *M* if *T* satisfies the following three axioms (Lipschutz 1965):

1. *M* and ϕ belong to *T*,
2. the union of any number of open sets in *T* belongs to *T*, and
3. the intersection of any two sets in *T* belongs to *T*.

The members of *T* are open sets and the space (*M*, *T*) is called topological space.

Limit Points

Let *M* be a topological space. A point $p \in M$ is a limit point of a subset *S* of *M* iff every open set *O* containing *p* contains a point of *S* different from *p* i.e.,

$$p \in O \Rightarrow (O - \{p\}) \cap S \neq \varphi .$$

Closure of Set

Let *S* be subset of a topological space *M*, then the closure of *S* is the intersection of all closed supersets of *S* and is denoted by $\overline{S}$.

Interior, Exterior and Boundary

Let *S* be subset of a topological space *M*. A point $p \in S$ is an interior point of *S* if $p \in O \subset S$, where *O* is an open set. The set of interior points of *S* is denoted by int(*S*) and is called the interior of *S*.

The exterior of *S* is the interior of complement of *S*, i.e., ext(S) = int(S^c).

The boundary of *S* is the set of all points which do not belong to the interior or the exterior of *S* and is denoted by $\dot{S}$, hence $\overline{S} = \text{int}(S) \cup \dot{S}$.

Neighborhoods

Let *p* be a point in a topological space *M*. A subset *N* of *M* is a neighborhood of *p* iff *N* is a superset of an open set *O* containing *p*, i.e., $p \in O \subset N$.

DIFFERENTIAL MANIFOLD

A locally Euclidean space is a topological space *M* such that each point has a neighborhood homeomorphic to an open subset of the Euclidean space R^n . A manifold is essentially a space which is locally similar to Euclidean space in that it can be covered by coordinate patches but which need not be Euclidean globally. A real scalar function on a differentiable manifold *M* is a map, $F : M \to R$ that assigns a real number $f(p)$ to every point $p \in M$ of the manifold. Map $\phi : O \to O'$ where $O \subset R^n$ and $O' \subset R^m$ is said to be a class $C^r (r \geq 0)$ if the following conditions are satisfied. If we choose a point *p* of coordinates $(x^1,..., x^n)$ on *O* and its image $\phi(p)$ of coordinates $(x'^1,..., x'^n)$ on O' then by C^r map we mean that the function ϕ is *r*-times differential and continuous. If a map is C^r for all $r \geq 0$ then we denote it by C^∞ ; also by C^0 map we mean that the map is continuous. This means that we can compare a manifold as smooth space (Hawking and Ellis 1973). Hence by identifying an open subset of a manifold with an open subset of R^n , the notion of differentiability of a function from R^n to R^m is passed on to one of a function from one manifold to another.

A differentiable manifold is roughly a smooth topological space, which locally looks like R^n. An n-dimensional, C^r, real differentiable manifold M is defined as follows (Mohajan 2013a):

A topological space M has a C^r altas $\{U_\alpha, \phi_\alpha\}$ where U_α are subsets of M and ϕ_α are one-one maps of the corresponding U_α to open sets in R^n (i.e., ϕ_α is a homeomorphism from U_α to an open subset of R^n) such that (figure 1);

i. U_α cover M, i.e., $M = \bigcup_\alpha U_\alpha$.

ii. If $U_\alpha \cap U_\beta \neq \phi$ then the map $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is a C^r map of an open subset of R^n to an open subset of R^n.

Condition (ii) is very important for overlapping of two local coordinate neighborhoods. Now suppose U_α and U_β overlap and there is a point p in $U_\alpha \cap U_\beta$. Now choose a point q in $\phi_\alpha(U_\alpha)$ and a point r in $\phi_\beta(U_\beta)$. Now $\phi_\beta^{-1}(r) = p$, $\phi_\alpha(p) = (\phi_\alpha \circ \phi_\beta^{-1})(r) = q$. Let coordinates of q be $(x^1, \ldots, x^n)$ and those of r be $(y^1, \ldots, y^n)$. At this stage we obtain a coordinate transformation;

$$y^1 = y^1(x^1, \ldots, x^n)$$

$$y^2 = y^2(x^1, \ldots, x^n)$$

$$\ldots \quad \ldots \quad \ldots$$

$$y^n = y^n(x^1, \ldots, x^n).$$

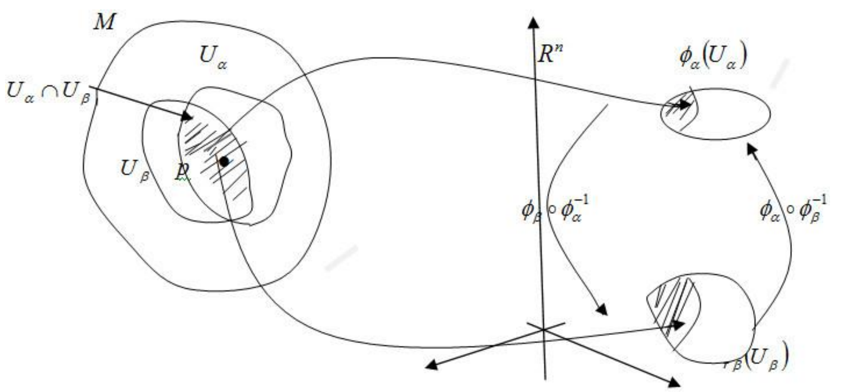

Figure 1: The smooth maps $\phi_\alpha \circ \phi_\beta^{-1}$ on the n-dimensional Euclidean space R^n giving the change of coordinates in the overlap region.

Given a local chart (U_α, ϕ_α) one can define physical quantities on U_α much like one would do on R^n, where the ϕ_α define coordinates on U_α. The different patches U_α can however be glued together in a non-trivial way by the transition functions $\phi_\beta \circ \phi_\alpha^{-1}$, so that globally a manifold is a generalization of R^n.

The open sets U_α, U_β and maps $\phi_\alpha \circ \phi_\beta^{-1}$ and $\phi_\beta \circ \phi_\alpha^{-1}$ are all *n*-dimensional, so that C^r manifold *M* is *r*-times differentiable and continuous i.e., *M* is a differentiable manifold. So that, whenever we will say manifold, we will mean differentiable manifold. General relativity is founded on the concept of differentiable manifolds. The mathematical model of space-time is given by a pair (M, g) where *M* is a differentiable manifold of dimension 4 and *g* is a metric that is a rule to calculate the length of curves connecting points of *M*.

Here we will discuss some definitions related to differential geometry following Mohajan (2013b) and Joshi (1996).

Hausdorff Space

A topological space *M* is a Hausdorff space if for pair of distinct points $p, q \in M$ there are disjoint open sets U_α and U_β in *M* such that $p \in U_\alpha$ and $q \in U_\beta$.

Paracompact Space

An atlas $\{U_\alpha, \phi_\alpha\}$ is called locally finite if there is an open set containing every $p \in M$ which intersects only a finite number of the sets U_α. A manifold *M* is called a paracompact if for every atlas there is locally finite atlas $\{O_\beta, \psi_\beta\}$ with each O_β contained in some U_α. Let V^μ be a timelike vector, then paracompactness of manifold *M* implies that there is a smooth positive definite Riemann metric $K_{\mu\nu}$ defined on *M*.

Homeomorphism

Two topological spaces M_1 and M_2 are called homeomophic if there exists a one-one onto function $f : M_1 \to M_2$ such that *f* and f^{-1} are C^0 continuous. The function *f* is called a homeomorphism. If *f* and f^{-1} are both C^r map then *f* is called C^r diffeomorphism. Homeomorphisms preserve all topological properties.

Tangent Space

A C^k-curve in *M* is a map from an interval of *R* in to *M* (figure 2). A vector $\left(\partial/\partial t\right)_{\lambda(t_0)}$ which is tangent to a C^1-curve $\lambda(t)$ at a point $\lambda(t_0)$ is an operator from the space of all smooth functions on *M* into *R* and is denoted by;

$$\left(\partial/\partial t\right)_{\lambda(t_0)}(f)=\left(\partial f/\partial t\right)_{\lambda(t_0)}=\lim_{s\to 0}\frac{f[\lambda(t+s)]-f[\lambda(t)]}{s},$$

where f be a function from M into R. If $\{x^i\}$ are local coordinates in a neighborhood of $p=\lambda(t_0)$ then;

$$\left(\partial f/\partial t\right)_{\lambda(t_0)}=\frac{dx^i}{dt}\cdot\frac{\partial f}{\partial x^i}\bigg|_{\lambda(t_0)}.$$

A vector X at the point p tangent to the curve is called a tangent vector to M at p. If $\{x^i\}$ are a set of coordinates on U_α, X can be represented by the components,

$X^\alpha=\frac{d}{dt}x^\alpha(\gamma(t))\Big|_{t=0}$. We define a tangent vector by the relation;

$$X(f)=\frac{d}{d}f(\lambda(t))\Big|_{t=0}$$

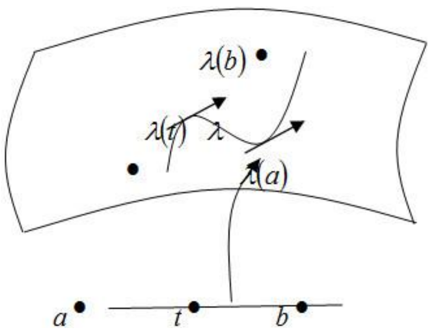

Figure 2: A curve in a differential manifold.

where X is represented by a differential operator, $X=X^\alpha\frac{\partial}{\partial x^\alpha}$. Hence the set $\left\{\frac{\partial}{\partial x^\alpha}\right\}$ can be considered as a basis and every tangent vector at $p\in M$ can be expressed as a linear combination of the coordinates derivates, $\left(\partial/\partial x^1\right)_p,\ldots,\left(\partial/\partial x^n\right)_p$. Thus the vectors $\left(\partial/\partial x^i\right)$ span the vector space T_p. Then the vector space structure is defined by $(aX+bY)f=a(Xf)+b(Yf)$, where $a,b\in R$ and pointwise multiplication is defined by $(f.g)(X)\equiv f(X)g(X)$. The vector space T_p is also called the tangent space at the point p, i.e., the collection of the vectors at p tangent to all curves that go through p is called the tangent space T_pM at p. Hence the tangent space T_pM to the manifold M in the point p is the vector space of first order differential operators on the smooth functions

$C^{\infty}(M)$ at *p*. A smooth assignment of a tangent vector at every point $p \in M$ is called a vector field $X(p)$ on *M*.

Orientation

In mathematics, orientability is a property of surfaces in Euclidean space measuring whether it is possible to make a consistent choice of surface normal vector at every point. Let *B* be the set of all ordered basis $\{e_i\}$ for T_p, the tangent space at point *p*. If $\{e_i\}$ and $\{e_j\}$ are in *B*, then we have $e_{j'} = a^i_{j'} e_i$. If we denote the matrix (a_{ij}) then $\det(a) \neq 0$. An *n*-dimensional manifold *M* is called orientable if *M* admits an atlas $\{U_i, \varphi_i\}$ such that whenever $U_i \cap U_j \neq \varphi$ then the Jacibian, $J = \det\left(\frac{\partial x^i}{\partial x^j}\right) > 0$, where $\{x^i\}$ and $\{x^j\}$ are local coordinates in U_i and U_j respectively. The M$\ddot{o}$bious strip is a non-orientable manifold (discuss later). A vector defined at a point in M$\ddot{o}$bious strip with a positive orientation comes back with a reversed orientation in negative direction when it traverses along the strip to come back to the same point.

Orientation of a Manifold

An orientation of a manifold is a choice of a maximal atlas, such that the coordinate changes are orientation preserving. An oriented atlas is called maximal if it cannot be enlarged to an oriented atlas by adding another chart. A topological manifold *M* is called orientable if it has a topological orientation, otherwise it is called non-orientable. For zero dimensional manifolds an orientation is a map $\varepsilon : M \to \{\pm 1\}$ (Kreck 2013).

An atlas $\{U_\alpha, \phi_\alpha : U_\alpha \to U_\beta \subset R^n\}$ is called oriented if all coordinate changes $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ are orientation preserving. A homeomorphism $f : N \to M$ between oriented topological manifolds is orientation preserving if for each chart $\phi : U \to V \subset R^n$ in the oriented atlas of *N* the chart ϕf is in the oriented atlas of *M*.

A real vector bundle, which a priori has a $GL(n)$ structure group, is called orientable when the structure group may be reduced to $GL^+(n)$, the group of matrices with positive determinant. For the tangent bundle, this reduction is always possible if the underlying base manifold is orientable and in fact this provides a convenient way to define the orientability of a smooth real manifold.

Space-time Manifold

In general relativity each point is an event so that coordinates specify not only it is where but also it is when. General relativity models the physical universe as a four-dimensional C^{∞} Hausdorff differentiable space-time manifold *M* of 4-dimentional with a Lorentzian metric *g* of signature (–,+,+,+) which is topologically connected, paracompact and space-time orientable. These properties are suitable when we consider for local physics. As soon as we investigate global features then we face various pathological difficulties such as, the violation of time orientation, possible non-Hausdorff or non-papacompactness, disconnected components of space-time etc. Such pathologies are to be ruled out by means of reasonable topological assumptions only. However, we like to ensure that the space-time is causally well-behaved. We will consider the space-time manifold (*M*, *g*) which has no boundary. By the word "boundary' we mean the 'edge' of the universe which is not detected by any astronomical observations. It is common to have manifolds without boundary; for example, for two-spheres S^2 in R^3 no point in S^2 is a boundary point in the induced topology on the same implied by the natural topology on R^3. All the neighborhoods of any $p \in S^2$ will be contained within S^2 in this induced topology. We shall assume *M* to be connected i.e., one cannot have $M = X \cup Y$, where *X* and *Y* are two open sets such that $X \cap Y \neq \varphi$. This is because disconnected components of the universe cannot interact by means of any signal and the observations are confined to the connected component wherein the observer is situated. It is not known if *M* is simply connected or multiply connected. *M* assumed to be Hausdorff, which ensures the uniqueness of limits of convergent sequences and incorporates our intuitive notion of distinct space-time events.

One-form

One-form is defined as linear, real valued function of vectors. We define a general basis by $\{e_i\}$ for *i* = 1,2,..,*n* which are linearly independent vectors. Then any vector $V \in T_p$ we can write,

$$V = V^i e_i$$

where the quantities V^i are called the components of *V* with respect to the basis e_i. In the coordinate basis we have $V^i = \frac{dx^i}{dt}$. Also we know $\left(\partial / \partial x_i \right)$ forms a basis of T_p at *p*, we can define the vector space of all the dual vectors at *p* is called covariant vectors or one-forms at *p*. A one-form ω at *p* is a real-valued linear function of T_p and is denoted by $\omega(X) \equiv X(\omega)\langle \omega, X \rangle$, where the last expression emphasizes the equal status of ω and *X*. Here $\omega(X)$ is often called the contraction of ω with *X*. In tensor algebra vectors are called contravariant vectors and one-forms are called covariant vectors. The linearity of one-form means (Hawking and Ellis 1973, Joshi 1996);

$$\langle \omega, aX + bY \rangle = a\langle \omega, X \rangle + b\langle \omega, Y \rangle$$

for $a, b \in R$ and $X, Y \in T_p$. Multiplication of a one-form by a real number *a* implies;

$$a\langle \omega, X \rangle = \langle a, \omega \rangle X \ .$$

Again for all *X*, $\omega + \sigma$ is the one-form such that;

$$\langle \omega + \sigma, X \rangle = \langle \omega, X \rangle + \langle \sigma, X \rangle .$$

The linear combinations of one-forms are defined by;

$$\langle a\omega + b\sigma, X\rangle = a\langle \omega, X\rangle + b\langle \sigma, X\rangle$$

for all X. We observe that one-forms at the point p satisfy the axioms of vector space, which is called the dual vector of T_p and is denoted by T_p^*. Given a tangent space basis $\{e_i\}$, a unique set of one-forms $\{e_i\}$ is defined by the condition that the given one-form e^j maps a vector V into V^j. Hence,

$\langle e^j, V\rangle = V^j$ and $\langle e^i, e_j\rangle = \delta^i_j$.

We can write the one-form ω as;

$\omega = \omega_i e^i$ i.e., $\omega_i = \langle \omega, e_i\rangle$.

For any $\omega \in T_p^*$ and $V \in T_p$ we can write,

$$\langle \omega, V\rangle = \langle \omega_i e^i, V^j e_j\rangle = \omega_i V^j \delta^i_j = \omega_i V^i .$$

Any smooth function f on M defines a one-form df, is called the differential of f as;

$$\langle df, V\rangle \equiv Vf .$$

Hence in a coordinate basis we have,

$$\langle df, V\rangle \equiv V^i \frac{\partial f}{\partial x^i} .$$

The local coordinate functions $(x^1,\dots, x^n)$ can be used to define a set of one-forms $(dx^1,\dots, dx^n)$, which gives a basis dual to the coordinate basis. Now we can write; $\left\langle dx^i, \frac{\partial}{\partial x^j}\right\rangle = \frac{\partial x^i}{\partial x^j} = \delta^i_j$, which gives,

$$df = \left\langle df, \frac{\partial}{\partial x^a}\right\rangle dx^a = \frac{\partial f}{\partial x^a} dx^a .$$

If f is non-constant function then the surface f = constant, define an $(n-1)$-dimensional sub-manifold M. For the set of all the vectors $V \in T_p$, such that, $\langle df, V\rangle = Vf = 0$, then the vectors V are tangent to curves in the f = constant sub-manifold through p. In such a situation df is normal to the surface f = constant at p.

We can write an arbitrary tangent vector as;

$$\frac{d}{d\lambda} = \sum_i \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i} .$$

So that the gradient df is defined by,

$$df\left(\frac{d}{d\lambda}\right)=\frac{df}{d\lambda}.$$

Hence we can write;

$$df\left(a\frac{d}{d\lambda}+b\frac{d}{d\mu}\right)=\left(a\frac{d}{d\lambda}+b\frac{d}{d\mu}\right)f=a\frac{df}{d\lambda}+b\frac{df}{d\mu}=af\left(\frac{d}{d\lambda}\right)+b\left(\frac{d}{d\mu}\right).$$

Hypersurface

In the Minkowski space-time $ds^2=-dt^2+dx^2+dy^2+dz^2$, the surface *t* = 0 is a 3-dimensional surface with the time direction always normal to it. Any other surface *t* = constant is also a spacelike surface in this sense. Let *S* be an $(n-1)$-dimensional manifold. If there exists a C^∞ map $\phi:S\to M$ which is locally one-one i.e., if there is a neighborhood *N* for every $p\in S$ such that ϕ restricted to *N* is one-one, and ϕ^{-1} is a C^∞ as defined on $\phi(N)$, then $\phi(S)$ is called an embedded sub-manifold of *M*. A hypersurface *S* of any *n*-dimensional manifold *M* is defined as an $(n-1)$-dimensional embedded sub-manifold of *M*. Let V_p be the $(n-1)$-dimensional subspace of T_p of the vectors tangent to *S* at any $p\in S$ from which follows that there exists a unique vector $n^a\in T_p$ and is orthogonal to all the vectors in V_p. n^a is called the normal to *S* at *p*. If the magnitude of n^a is either positive or negative at all points of *S* without changing the sign, then n^a could be normalized so that $g_{ab}n^a n^b=\pm 1$. If $g_{ab}n^a n^b=-1$ then the normal vector is timelike everywhere and *S* is called a spacelike hypersurface. If the normal is spacelike everywhere on *S* with a positive magnitude, *S* is called a timelike hypersurface. Finally, *S* is null hypersurface if the normal n^a is null at *S*.

Lie Algebra

The set of the infinitesimal generators $\Lambda=\{X\}$ is a linear algebra on the field *K* where the group transformations are defined and the Lie product of two operators is simply their commutator (Gourdin 1982).

$$[\alpha X+\beta Y,Z]=\alpha[X,Z]+\beta[Y,Z]$$

$$[X,\alpha Y+\beta Z]=\alpha[X,Y]+\beta[X,Z]$$

$\forall\alpha,\beta\in K$ and $X,Y,Z\in\Lambda$.

A Lie Algebra is a linear algebra which satisfies the anti-symmetry property;

$$[X,X]=0,\forall X\in\Lambda.$$

The fundamental relation of Lie Algebra is the following:

$$[X_{\alpha,}X_\beta]=C^{\gamma}_{\alpha\beta}X_\gamma$$

where the quantities $C^{\gamma}_{\alpha\beta}$ are called the structure constant of the Lie Algebra.

The Jacobian identity is,

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

For infinitesimal generators this gives the relation for the structure constants. The general linear group $GL(n,R)$ is defined as the set of regular linear transformations of R^n. An evident complete basis of $n\times n$ matrices is obtained with matrices having only one non-vanishing element. Let us choose for convenience, $[E_{rs}]_{mn}= g_{mr}g_{ns}$ and we have n^2 such matrices. The matrices E_{rs} are a matrix representation of the Lie algebra of $GL(n,R)$ and the commutation relations have the explicit form;

$$[E_{rs},E_{tu}]= g_{at}E_{ru}-g_{ur}E_{ts}.$$

FIBRE BUNDLE

Fibre bundle is absolutely central in modern physics which provides the appropriate mathematical framework to formulate modern field theory. There are two kinds of fibre-bundles: principal bundles and associated bundles. The notion of a principal fibre-bundle is the appropriate mathematical concept underlying the formulation of gauge theories. Gauge theories describe the dynamics of all non-gravitational interactions (interactions of photon, gluon, graviton etc. bosons). The associated fibre-bundles provide mathematical framework to describe matter fields that interact through the exchange of the gauge bosons.

Basic Concepts of Fibre Bundle

Fibre bundle is a very interesting manifold and is formed by combining a manifold *M* with all its tangent spaces T_p. A fibre bundle is a manifold that looks locally like a product of two manifolds, but is not necessarily a product globally. A bundle whose fibre is a one-dimensional vector space is called a line bundle. A fibre bundle will be called trivial if it can be described as a global product. Because of the importance of fibre bundles in modern theoretical physics, many introductory expositions of fibre bundles for physicists exist. For simplicity let us consider a one-dimensional manifold *M* (a curve) and its tangent spaces. Figure 3a shows a curve *M* and a few tangent spaces which are straight lines drawn tangent to the curve, and each must be thought of as extending infinitely far in both directions. When we draw tangents in such a process the picture of course will be messy due to large number of tangent spaces intersecting one another and leaving the curve *M* haphazardly. So, we look for a better way like figure 3b where the tangent spaces are drawn parallel, they cross *M* only at the point

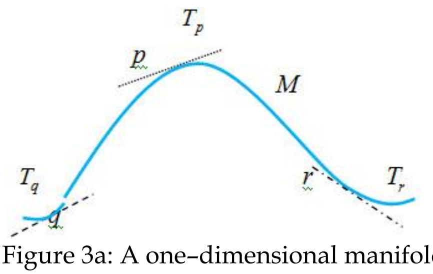

Figure 3a: A one–dimensional manifold and some of its tangent spaces.

where they are defined. Here T_p represents not tangent to the curve but vector at each *p*. So, we define new manifold *TM*, consisting of all vectors at all points, which is two-dimensional. It is called a fibre bundle where the fibres are the spaces T_p for each *p*. A general fibre bundle consists of a base manifold, which is the curve *M* and one fibre attached to each point of the base space. If the base space is *n*-dimensional and each fibre is *m*-dimensional then the bundle has

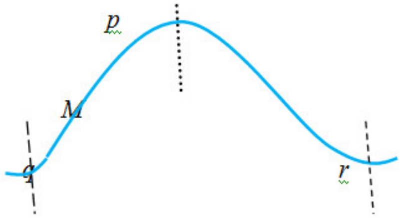

Figure 3b: Same as figure 3a but here tangent spaces are drawn parallel to one another to avoid spurious intersections.

$(m+n)$-dimensions. The points of a single fibre are related to one another while points on different fibres are not. This is formalized by defining a projection map π, which maps any point of a fibre to the base manifold defined on it (Schutz 1980). Hence a fibre bundle is a manifold *M* with a copy of the fibre *F* at every point of *M*.

Now we shall study the global properties of the fibre bundles. For simplicity we consider the product space. Two spaces *M* and *N* have Cartesian product space $M \times N$ consisting of all ordered pairs (*a*,*b*) where $a \in M$ and $b \in N$. If *M* and *N* are manifolds, $M \times N$ is also a manifold. The set of coordinates $\{x^i, i = 1,..., m\}$ of an open set *U* of *M* taken together with open set *V* of *N*, from a set of $m+n$ coordinates for the open set (U,V) of $M \times N$. The fibre bundles defined above, at least locally, product spaces, the product $(U \times F)$ of open set *U* of the base manifold *B* with the space *F* representing a typical fibre. It is locally trivial but globally it is not trivial. The above property is defined as follows (Schutz 1980):

Let us consider TS^1, the tangent bundle of the circle S^1. TS^1 is identical to the product space $S^1 \times R$, as shown in figure 4a which is global version of the local picture as shown in figure 3b.

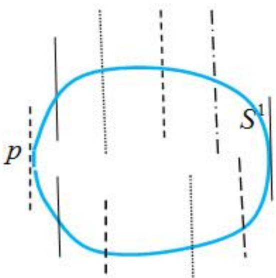

Figure 4a: The trivial way of constructing TS^1 as the product space of the circle S^1 and the typical fibre R^1.

If we cut the circle at point y and unwrap the bundle, laying it as figure 4b.

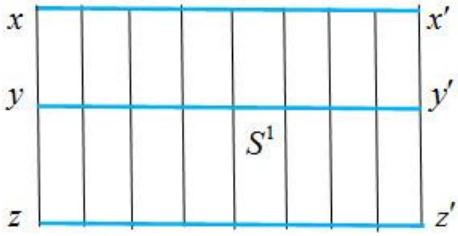

Figure 4b: TS^1 cut along one fibre and laid flat. Each fibre is extended infinitely far in the vertical direction.

The Mö bius Strip

To form a Mö bius strip we consider a rectangular strip. It is of course be seen as the product of two line segments. If we want to join two opposite edges of the strip to turn one of the line segments into a circle, there are two ways to do this. To reconstruct figure 4a from figure 4b we simply identify point x with x', y with y' and z with z', and so on. Joining of the two edges in a straightforward way we can form a cylinder C (Figure 5). It should be of course clear that the cylinder is not only locally a product, but also globally; namely $C = S^1 \times L$, where L is a line segment which is not only locally a product but also globally. The cylinder is a global diffeomorphism from $S^1 \times L$ to C (Nash and Sen1983, Nakahara 1990).

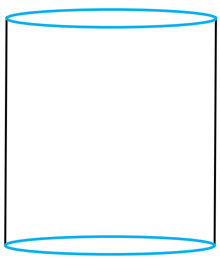

Figure 5: The cylinder version of the fibre bundle.

But in a different way we can form a Mö bius band as follows: Identify x with z', y with y' and z with x', and so on. This is a twist so that it looks like figure 6 when joined together. Locally it is still the same as figure 4a. In fact the bundle over any connected open proper subset of S^1

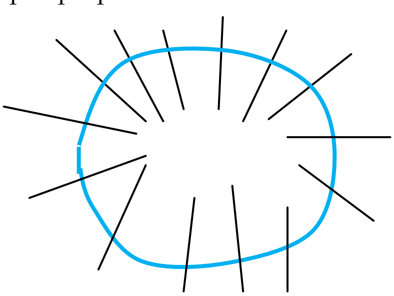

Figure 6: Mö bius band version of the fibre bundle.

has one-one continuous map into the same portion of figure 4a. Locally, along each open subset U of the S^1, the Möbius strip, M_o still looks like a product, $M_o = U \times L$. However, globally there is no unambiguous and continuous way to write a point m of M_o as a Cartesian pair $(s,t) \in S^1 \times L$. Locally, $\pi : M_o \to S^1$, hence for every $x \in S^1$, its inverse image is isomorphic to the line segment, $\pi^{-1}(x) = L$. For every open subset U of S^1, we can define a diffeomorphism $\phi : U \times L \to \pi^{-1}(U)$, i.e., for every element p of $\pi^{-1}(U) \subset M_o$, we can assign local coordinates $\phi^{-1}(p) = (x,t)$, where $x = \pi(p) \in U$ and $t \in L$.

Möbius strip covers the circle by two open sets, U_1 and U_2, which overlap on two disjoint open intervals, A and B (Collinucci and Wijns 2006). Here we have the two maps (Figure 7);

$$\phi_1 : U_1 \times L \to \pi^{-1}(U_1)$$
$$\phi_2 : U_2 \times L \to \pi^{-1}(U_2)$$

Here $U_1 \cup U_2$ covers M_o and ϕ_1 and ϕ_2 are homeomorphisms h_{12} which define an atlas for M_o from L to L in such a way that $\phi_1^{-1}\phi_2(x,t) = (x, h_{12}(t))$. So, Möbius band is not a product space globally that is, of a non-trivial fibre bundle.

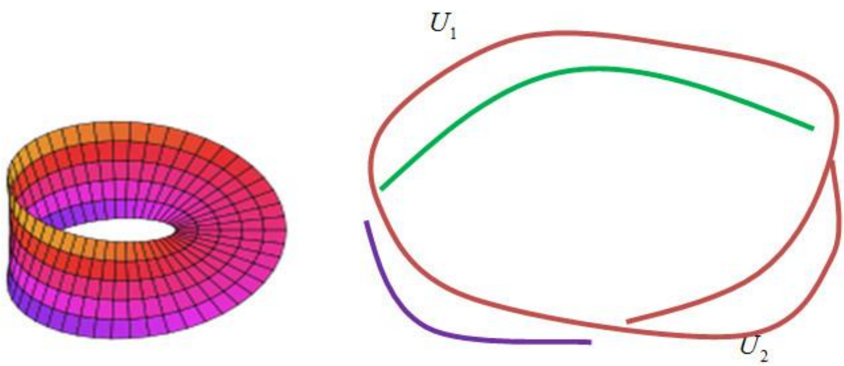

Figure 7: Möbius band is not a product space globally.

The difference between above two bundles over S^1 is in what is called the bundles 'structure group'. We define a fibre bundle as a space E for which the following are given a base manifold B, a projection $\pi : E \to B$, a typical fibre F, a structure group G of homeomorphism of F into itself, and a family $\{U_j\}$ of open sets covering B, all of which satisfies the following restrictions:
The bundle over any set U_i, which is $\pi^{-1}(U_i)$, has a homeomorphism onto the product space $U_i \times F$ i.e., locally the bundle is trivial. Part of this homeomorphism is a homeomorphism from each fibre, say $\pi^{-1}(x)$ where x is an element of B, onto F. Let us call this map $h_i(x)$.

When two sets U_i and U_j overlap, a given point x in their intersection has two homeomorphism $h_i(x)$ and $h_j(x)$ from its fibre onto F. Since a homeomorphism is invertible, the map $h_i(x) \circ h_j^{-1}(x)$ is a homeomorphism of F onto F which is necessary to be an element of the structure group G.

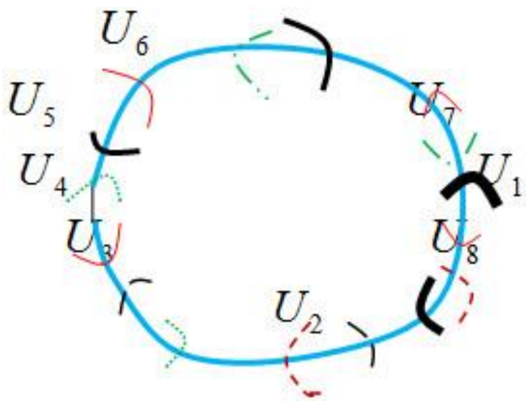

Figure 8: A set of neighborhoods of S^1 which cover S^1. The extent of each neighborhood is indicated by the parentheses, U_1 overlaps U_2, and so on until U_8 overlaps U_1.

The second restriction is global structure of the fibre bundle. To check this, we first introduce the complete definition of TS^1. The bundle $E = TS^1$ has base $B = S^1$, typical fibre $F = R^1$, and projection $\pi : (x, v) \to x$ where x is a point of S^1 and v is a vector in T_x. Let the covering $\{U_i\}$ be the open sets of any atlas of S^1, (figure 8). Every U_i has a coordinate system i.e., a parametrization of S^1, which we will call λ_i. The vector $d/d\lambda_i$ at x in U_i is a basis for T_x, so, any vector v in T_x has representation $\alpha_{(i)} d/d\lambda_i$ for any index i, and $\alpha_{(i)} \in R$.

The homeomorphism of T_x onto R which are part of the definition of TS^1 are defined to be $h_i(x) : v \to \alpha_{(i)}$. If x belongs to two neighborhoods U_i and U_j there are two such homeomorphisms from T_x on to R, and since λ_i and λ_j are unaltered, $\alpha_{(i)}$ and $\alpha_{(i)}$ can be any two real numbers. The homeomorphism $h_i(x) \circ h_j^{-1}(x) : F \to F$ maps $\alpha_{(i)} \to \alpha_{(j)}$ and is therefore just multiplication by the number $r_{ij} = \alpha_{(i)} / \alpha_{(j)}$. Since r_{ij} is any real number other than zero, the structure group is $R^1 - \{0\}$, which is a Lie Group. For an n-dimensional manifold M the structure group of TM is the set of all $n \times n$ matrices with non-zero determinant, which is called GL (n, R) which defines TS^1.

To characterize the structure of the M$\ddot{o}$bius band we must use different maps $h_i(x)$. Let us use the family $\{U_i, i = 1,2,...,8\}$ and also define $r_{12} = 1, r_{23} = 1,..., r_{78} = 1$. But in this case the twist in the M$\ddot{o}$ bius band faces us to use $r_{81} = -1$. The structure group here is a multiplicative group with elements $\{1,-1\}$. The tangent bundle TS^1 has the structure group is $R^1 - \{0\}$, which is nearly the same as its typical fibre. The frame bundle of any manifold *M* has the same structure group as *TM*, but its fibre is the set of all bases for the tangent space.

In the case of a one-dimensional manifold like S^1, this is the set of all non-zero vectors, which is identical to $R^1 - \{0\}$. So, the fibre bundle of has fibres homeomorphic to its structure group, and this is true for all frame bundles. Such a group is called a principal fibre bundle (Schutz 1980).

Corollary: The M$\ddot{o}$ bious band is a non-orientable manifold.

Proof: The center circle in a M$\ddot{o}$ bious band, M_o is an orientable sub-manifold, but it does not admit a continuous vector field which is nowhere tangent to it. Hence the M_o cannot be orientable.

Definition of a Fibre Bundle

A differentiable fibre bundle (E, π, M, F, G) consists of the following elements (Collinucci and Wijns 2006):

i) A differentiable manifold *E* is called the total space.
ii) A differentiable manifold *M* is called the base space.
iii) A differentiable manifold *F* is called the typical fibre.
iv) A surjection $\pi : E \to M$ is called the projection map which is smooth. For $x \in M$, the inverse image $\pi^{-1}(x) \equiv F_x \cong F$ is called the fibre at *x*.
v) A (Lie) Group *G* is called the structure group, which acts on the fibre on the left and it is called principal bundle. A principal bundle can be thought of the parent of various associated bundles, which are constructed by allowing the Lie group to act on a fibre.
vi) An open covering $\{U_i\}$ of *M* and a set of diffeomorphisms $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi\phi_i(x,t) = x$. The map ϕ_i is called a local trivialization.
vii) At each point $x \in M$, $\phi_{i,x}(t) \equiv \phi_i(x,t)$ is a diffeomorphism, $\phi_{i,x} : F \to F_x$. On each overlap $U_i \cap U_j \neq \varphi$, we require $h_{ij} = \phi_{i,x}^{-1}\phi_{j,x} : F \to F$ to be an element of *G*, i.e., we have a smooth map $h_{ij} : U_i \cap U_j \to G$ such that $\phi_j(x,t) = \phi_i(x, h_{ij}(x)t)$.

Mathematically this defines a coordinate bundle. From (vi) it follows that $\pi^{-1}(U_i)$ is diffeomorphic to a product, the diffeomorphism is given by $\phi_i^{-1} : \pi^{-1}(U_i) \to U_i \times F$. It is in this sense that *E* is locally a product. We usually require that all fibres be diffeomorphic to some fixed manifold *F*. A bundle that is a product $E = M \times F$, is said to be trivial (Frankel 1999).

Principal Bundles and Associated Vector Bundles

The collection of all tangent vectors to a manifold *M* at a point *p* is a vector space called the tangent space T_pM. The collection $\{T_pM | x \in M\}$ of all tangent spaces of *M* is called the tangent bundle *TM*. Its base manifold is *M* and fibre is R^n, where *n* is the dimension of *M*, and its structure group is a subgroup of $GL(n,R)$. If *M* is R^n, the tangent space to every point is isomorphic to *M* itself. Its tangent bundle TR^n is trivial and equal to $R^n \times R^n \cong R^{2n}$. The circle is not contractible, yet its tangent bundle TS^1 is trivial. The tangent bundle of the 2-sphere TS^2 is a nontrivial bundle. There is no global diffeomorphism between TS^2 and $S^2 \times R^2$, since in this case one cannot even find a single global non-vanishing vector field and one would have to be able to define two linearly independent vectors at every point of the sphere in a smooth fashion. A set of pointwise linearly independent vectors over an open set of the base manifold of a tangent bundle is called a frame FS^2. We have seen that TS^2 is non-trivial globally, because we cannot able to find a frame over the entire sphere. At each point of frame we can of course construct many different sets of linearly independent vectors. The set of all possible frames over an open set *U* of S^2 is diffeomorphic to $U \times GL(2,R)$. Globally this becomes a bundle over S^2 with fibre $GL(2,R)$ and is called the frame bundle FS^2 of S^2 (Collinucci and Wijns 2006). An associated vector bundle is a fibre-bundle where the standard fibre *F* = *V* is a vector space and the action of the structural group on the standard fibre is a linear representation of *G* on *V*. A tangent bundle is always a vector bundle. The Möbius strip is not a vector bundle. A principal bundle has a fibre which is identical to the structure group *G*. It is usually denoted by *P*(*M*, *G*) and called a *G*-bundle over *M*.

CONCLUDING REMARKS

In this study we have discussed preliminary ideas of differential manifold and fibre bundles. We have discussed some definitions to make the paper interesting to the readers. Throughout the study we have avoided difficult mathematical calculations. The paper will be helpful for those readers who need very elementary idea of differential geometry.

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